Problem 1

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty} \frac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent by the Test for Divergence.

Problem 2

$$a)\lim_{n\to\infty}\left\{\left(\sin\frac{1}{n^3}\right)/\left(\frac{1}{n^3}\right)\right\}=\lim_{n\to\infty}\frac{\frac{d}{dn}\left(\sin\frac{1}{n^3}\right)}{\frac{d}{dn}\left(\frac{1}{n^3}\right)}=\lim_{n\to\infty}\frac{\cos\frac{1}{n^3}\frac{d}{dn}\left(\frac{1}{n^3}\right)}{\frac{d}{dn}\left(\frac{1}{n^3}\right)}$$

l'Hôpital's rule

$$=\lim_{n\to\infty}\cos\frac{1}{n^3}=1>0$$

$$b)\ \sum_{n=1}^{+\infty}\frac{1}{n^3}\ \text{ is convergent p-series (p = 3 > 1)}$$

Taking into account the result from a) where $\lim_{n\to\infty}\left\{\left(\sin\frac{1}{n^3}\right)/\left(\frac{1}{n^3}\right)\right\}=1>0$

then the Limit comparison tests ensures also that

$\sum_{n=1}^{+\infty}\sin\frac{1}{n^3}$ is convergent

Problem 3

If $a_n = \dfrac{(5x-4)^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(5x-4)^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(5x-4)^n}\right| = \lim_{n\to\infty}|5x-4|\left(\frac{n}{n+1}\right)^3 = \lim_{n\to\infty}|5x-4|\left(\frac{1}{1+1/n}\right)^3$$

$$= |5x-4|\cdot 1 = |5x-4|$$

By the Ratio Test, $\sum_{n=1}^{\infty}\frac{(5x-4)^n}{n^3}$ converges when $|5x-4|<1 \quad\Leftrightarrow\quad \left|x-\frac{4}{5}\right|<\frac{1}{5} \quad\Leftrightarrow\quad -\frac{1}{5}<x-\frac{4}{5}<\frac{1}{5} \quad\Leftrightarrow$

$\frac{3}{5}<x<1$, so $R=\frac{1}{5}$. When $x=1$, the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series ($p=3>1$). When $x=\frac{3}{5}$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ converges by the Alternating Series Test.*

The series is convergent for 3/5≤x≤1

**You can also say that it is absolute convergent as a p-series with p=3 so it is convergent*

Problem 4

The series is a convergent geometric series so that

$$|e^c| = e^c < 1 \text{ thus we must have } c < 0$$

$$\sum_{n=0}^{+\infty} e^{nc} = \frac{1}{1-e^c} = D \Rightarrow 1-e^c = \frac{1}{D} \Rightarrow 1-\frac{1}{D} = e^c$$

$$c = \ln\left(1-\frac{1}{D}\right) < 0 \text{ as it should because D} > 1.$$

Problem 5

$$a)\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{4}{n^6}\frac{1}{(1+x^2)} = 0 \text{ for x} \in (-\infty,+\infty)$$

$b)$

$$|f_n(x)-0| = f_n(x) = \frac{4}{(x^2+1)n^6} < \frac{4}{n^6} < \frac{4}{N^6} = \varepsilon \quad \text{for n} > \text{N}$$

For every ε there is N (given by $\frac{4}{N^6} = \varepsilon$) so that for n > N to have $|f_n(x)-0| < \varepsilon$

for all $x \in R \Rightarrow$ Uniform convergent for $x \in R$

Problem-6

The differential equation is $mx'' + kx = F_0 \cos\omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$ with roots $\pm\sqrt{k/m}\,i = \pm\omega i$ so $x_c(t) = c_1 \cos\omega t + c_2 \sin\omega t$. Since $\omega_0 \neq \omega$, try $x_p(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

Then we need $(m)\left(-\omega_0^2\right)(A\cos\omega_0 t + B\sin\omega_0 t) + k(A\cos\omega_0 t + B\sin\omega_0 t) = F_0 \cos\omega_0 t$ or $A\left(k - m\omega_0^2\right) = F_0$ and $B\left(k - m\omega_0^2\right) = 0$. Hence $B = 0$ and $A = \dfrac{F_0}{k - m\omega_0^2} = \dfrac{F_0}{m(\omega^2 - \omega_0^2)}$ since $\omega^2 = \dfrac{k}{m}$. Thus the motion of the mass is given by $x(t) = c_1 \cos\omega t + c_2 \sin\omega t + \dfrac{F_0}{m(\omega^2 - \omega_0^2)} \cos\omega_0 t$.